Number Systems and Their Operations

(MATHEMATICS)

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PREFACE

The number systems have been developing rapidly to meet the demand of present day context in the digital world. It plays a vital role from usual counting to artificial intelligence counting. All the modern computer technologies are based on binary system and widely use octal and hexadecimal number systems.

In general the common peoples learn only the short cut technique of arithmetic operations (addition, multiplication, subtraction and division) of integers without knowing the actual rules of operations. Therefore, they find difficult to understand the arithmetic operations of other number systems in higher studies.

In this book effort is made to simplify the basic concepts of arithmetic operations in decimal, binary, octal and hexadecimal systems.

This book can be useful for the beginners of all disciplines to study mathematics and computer science. For basic conception it can also be treated as reference book of all standard from kindergarten to college.

I shall gratefully accept all suggestions, comments and criticisms for improvement of the book.

Dr. Abdul Wahed

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Chapter-1

Decimal Numbers

1.1 Number Systems:

A number system in mathematics is a well-defined system of representing, writing and expressing a numerical value which occurs in the measures of different dimensions and calculations. Numbers- the numeral values are different for different types of computing system. In mathematics basically there are four types of number systems.

1. Decimal Number System.
2. Binary Number System
3. Octal Number System and
4. Hexadecimal Number System.

1.2 Decimal Number System:

The decimal number system has base 10 and it uses the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to represent a number. A number is formed with single digit or more than one digit (two digits, three digits etc.). The decimal point (.) is used to separate the integral and fractional part of a decimal number. The successive places to the left of the decimal point are termed as unit, ten, hundred, thousand, and so on. The successive places to the right of the decimal point are termed as 10^{th}, 100^{th}, 1000^{th}, and so on.

Table 1.1: Place Value Chart for Decimal Numbers

Numbers											
100,000	10,000	1,000	100	10	1	.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$	$\frac{1}{10,000}$	$\frac{1}{100,000}$
Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones	Decimal Point	tenths	hundredths	thousandths	ten thousandths	hundred thousandths
HTH	TTh	Th	H	T	O	.	t	h	th	tth	hth
Symbols											

Table 1.2: Big and Small (fractional) Numbers in Decimal System (SI)

Name	Number	Expression in power of 10	Prefix	Symbol
yotta		10^{24}	yotta	Y
zetta		10^{21}	zetta	Z
exametres		10^{18}	exa	E
peta		10^{15}	peta	P
trillion	1,000,000,000,000	10^{12}	tera	T
billion	1,000,000,000	10^{9}	giga	G
million	1,000,000	10^{6}	mega	M
thousand	1,000	10^{3}	kilo	k
hundred	100	10^{2}	hecto	h
ten	10	10^{1}	deka	da
unit	1	10^{0}		
tenth	0.1	10^{-1}	deci	d
hundredth	0.01	10^{-2}	centi	c
thousandth	0.001	10^{-3}	milli	m
millionth	0.000001	10^{-6}	micro	μ
billionth	0.000000001	10^{-9}	nano	n
trillionth	0.000000000001	10^{-12}	pico	P
femto		10^{-15}	femto	f
atto		10^{-18}	atto	a
zepto		10^{-21}	zepto	z
yocto		10^{-24}	yocto	y

Table 1.3: Indian Numbering Units and Corresponding SI Units

Indian System		International System of Units (SI)		Expression in Power of 10
Numbers	Notations	Numbers	Notations	
Ten	10	Ten	10	10^1
Hundred	100	Hundred	100	10^2
Thousand	1,000	Thousand	1,000	10^3
Ten Thousand	10,000	Ten Thousand	10,000	10^4
Lakh (Lac or L)	1,00,000	Hundred Thousand	100,000	10^5
Ten Lakh	10,00,000	Million	1,000,000	10^6
Crore	1,00,00,000	Ten Million	10,000,000	10^7
Ten Crore	10,00,00,000	Hundred Million	100,000,000	10^8
Hundred Crore	1,00,00,00,000	billion	1,000,000,000	10^9

1.3 Roman Numerals:

The Roman Numeral System is a decimal (base-10) number system. But instead of place value in power of 10 the system uses a set of symbols with fixed value. The Roman Numeral System is originated in ancient Roman civilization. In this system the numbers are expressed by the combinations of letters from the Latin alphabet. It uses seven basic symbols, each associated with a fixed integral value. All other numerals are expressed with the help of these seven symbols. The small Roman numerals are generally used in writing serial numbers and large numerals are used to represent year number. Roman numerals are also used in clock.

Table 1.4

Basic Symbols	I	V	X	L	C	D	M
Integral Values	1	5	10	50	100	500	1000

Table 1.5: Roman Numerals (from 1 to 50)

Numbers		Numbers		Numbers		Numbers		Numbers	
Decimal	Roman	Decimal	Roman	Decimal	Roman	Decimal	Roman	Decimal	Roman
1	I	11	XI	21	XXI	31	XXXI	41	XLI
2	II	12	XII	22	XXII	32	XXXII	42	XLII
3	III	13	XIII	23	XXIII	33	XXXIII	43	XLIII
4	IV	14	XIV	24	XXIV	34	XXXIV	44	XLIV
5	V	15	XV	25	XXV	35	XXXV	45	XLV
6	VI	16	XVI	26	XXVI	36	XXXVI	46	XLVI
7	VII	17	XVII	27	XXVII	37	XXXVII	47	XLVII
8	VIII	18	XVIII	28	XXVIII	38	XXXVIII	48	XLVIII
9	IX	19	XIX	29	XXIX	39	XXXIX	49	XLIX
10	X	20	XX	30	XXX	40	XL	50	L

Table 1.6: Roman Numerals (from 51 to 100)

Numbers		Numbers		Numbers		Numbers		Numbers	
Decimal	Roman	Decimal	Roman	Decimal	Roman	Decimal	Roman	Decimal	Roman
51	LI	61	LXI	71	LXXI	81	LXXXI	91	XCI
52	LII	62	LXII	72	LXXII	82	LXXXII	92	XCII
53	LIII	63	LXIII	73	LXXIII	83	LXXXIII	93	XCIII
54	LIV	64	LXIV	74	LXXIV	84	LXXXIV	94	XCIV
55	LV	65	LXV	75	LXXV	85	LXXXV	95	XCV
56	LVI	66	LXVI	76	LXXVI	86	LXXXVI	96	XCVI
57	LVII	67	LXVII	77	LXXVII	87	LXXXVII	97	XCVII
58	LVIII	68	LXVIII	78	LXXVIII	88	LXXXVIII	98	XCVIII
59	LIX	69	LXIX	79	LXXIX	89	LXXXIX	99	XCIX
60	LX	70	LXX	80	LXXX	90	XC	100	C

1.3.1 Standard Form of Writing the Roman Numerals:

The following table 2.7 displays how the roman numerals are written in their respective individual decimal places.

Table 2.7: Roman Numerals in Individual Decimal Places

	Thousands		Hundreds		tens		Units	
	Form	Meaning	Form	Meaning	Form	Meaning	Form	Meaning
1	M	1 Thousand	C	1 Hundred	X	1 ten	I	One
2	MM	2 Thousands	CC	2 Hundreds	XX	2 tens	II	Two
3	MMM	3 thousands	CCC	3 Hundreds	XXX	3 tens	III	Three
4			CD	4 Hundreds	XL	4 tens	IV	Four
5			D	5 Hundreds	L	5 tens	V	Five
6			DC	6 Hundreds	LX	6 tens	VI	Six
7			DCC	7 Hundreds	LXX	7 tens	VII	Seven
8			DCCC	8 Hundreds	LXXX	8 tens	VIII	Eight
9			CM	9 Hundreds	XC	9 tens	IX	Nine

1.3.2 Techniques of writing the Roman numerals:

Roman numerals are written using addition and subtraction notation. After every three successive numbers it follows the subtractive notation. The process is repeated in succeeding steps.

a) Subtractive Notation: The possible groups of numerals in subtractive notation are displayed in thefollowing table 2.8. The smaller place value is subtracted from greater place value.

Table 2.8: Subtractive Notation:

Operation		Notation	Value
5−1	V−I	IV	4
10−1	X−I	IX	9
50 − 10	L−X	XL	40
100−10	C−X	XC	90
500−100	**D−C**	**CD**	**400**
1000−100	**M−C**	**CM**	**900**

b) Additive Notation: Smaller place value is added to a greater place value

Table 2.9: Additive Notation

Value added	Operation	Notation	Value
1		I	1
1+1	I+I	II	2
1+1+1	I+ I+I	III	3
…………			
5+1	V+I	VI	6
5+1+1	V+I+I	VII	7
5+1+1+1	V+ I+ I+I	VIII	8
………..			
10+1	X+I	XI	11
10+1+1	X+I+I	XII	12
10+1+1+1	X+ I+ I+I	XIII	13
……….			

1.3.3 Solved Examples:

29 = 20+9
= XX+IX
= XXIX

20022= 2000 +22
= (1000+1000) + (20+2)
= (M+M) + (XX+II)
=MMXXII

146 = 100 +40+ 6
= 100+(50 − 10) + 6
= C+(L − X) + VI
= CXLVI

950= 900+50
=(1000−100)+50
=(M−C)+L
=CML

475= 400 +70+ 5
= (500 − 100) +(50+ 10 + 10) + 5
= (D − C) + (L+ X + X) + V
= CDLXXV

649= 600 +40+ 9
= (500 + 100) +(50− 10) + (10 − 1)
= (D + C) + (L− X) + (X − I)
= DCXLIX

3500= 3000 +500
= (1000 + 1000 + 1000) + 500
= (M + M + M) + D = MMMD

Chapter-2

Arithmetic Operations

2.1 Decimal Operations:

The fundamental arithmetical operations of decimal numbers are addition (+), multiplication (×, .), subtraction (−) and division (÷ , *or* /) .These are illustrated step by step in this chapter.

2.2 Addition of decimal numbers:

Addition starts from unit place. The rules of addition of decimal numbers are illustrated in the table-3.1 for conceptual understanding.

First, let us take the two numbers 4879 and 3598 for addition (+).

4869
+7598
....
12467 (How the result is found?)

2.3 Carry Addition Techniques: (Description of the above example)

1. Addition in unit place (1st column) gives 17. The result is 7 and carry is 1 to the nextcolumn.
2. Addition in 2nd column gives 15. Now 15 +1 (carry) =16. The result is 6 and carry is 1 to the next column.
3. Addition in 3rd column gives 13. Now 13+1 (carry) =14. The result is 4 and carry is 1 to the next column.
4. Addition in 4th column gives 11. Now 11+1 (carry) =12. The result is 12.

2.4 Carry Addition Rules:

In addition when a number is carried from unit-place to 10th place then actual carry value is 10 times of the carried number. When a number is carried to 100th place then actual carry value is 100 times of the carried number and so on. The addition of 4869 + 7598 are illustrated using the actual addition rules as given in the following table 2.1 (carry numbers are displayed in *italic* and ***italic-bold*** font).

Table 2.1: Addition Rules

Place Values ⇒	10000th	1000th	100th	10th	Unit
Carry numbers ⇒	*10000*	*1000*	*100*	*10*	
4869=		4000	800	60	9
+7598=		7000	500	90	8
Column addition ⇒		12000 =*10000*+2000	1400 =*1000*+400	160 =*100*+60	17 =*10*+7
Results ⇒ 12467=	10000	2000	400	60	7

Table 2.2: Writing Techniques and Addition Rules of Decimal numbers:

Addition	Decimal Numbers	Carry number	Value of carry number
0+0=	0		
0+1=	1		
1+1=	2		
2+1=	3		
3+1=	4		
………	………		
8+1=	9		10
9+1=	10	0 in unit place carry 1 ten to 10th place	10
10+1=	11	1 in unit place carry 1 ten to 10th place	10
11+1=	12	2 in unit place carry 1 ten to 10th place	10
……..	………	………………	20
19+1=	20	0 in unit place carry 2 tens to 10th place	20
20+1=	21	1 in unit place carry 2 tens to 10th place	20
………	…….	…………	20
28+1=	29	9 in unit place carry 2 tens to 10th place	30
29+1=	30	0 in unit place carry 3 tens to 10th place	30
30+1=	31	1 in unit place carry 3 tens to 10th place	30
……..	……	……………	30
38+1=	39	9 in unit place carry 3 tens to 10th place	40
39+1=	40	0 in unit place carry 4 tens to 10th place	40
40+1=	41	1 in unit place carry 4 tens to 10th place	40
……..	……	……………	40
48+1=	49	9 in unit place carry 4 tens to 10th place	50
49+1=	50	0 in unit place carry 5 tens to 10th place	50
50+1=	51	1 in unit place carry 5 tens to 10th place	50
……..	……	……………	50
58+1=	59	9 in unit place carry 5 tens to 10th place	60
59+1=	60	0 in unit place carry 6 tens to 10th place	60
60+1=	61	1 in unit place carry 6 tens to 10th place	60
……..	……	……………	60
68+1=	69	9 in unit place carry 6 tens to 10th place	70
69+1=	70	0 in unit place carry 7 tens to 10th place	70
70+1=	71	1 in unit place carry 7 tens to 10th place	70
……..	……	……………	70
78+1=	79	9 in unit place carry 7 tens to 10th place	80
79+1=	80	0 in unit place carry 8 tens to 10th place	80
80+1=	81	1 in unit place carry 8 tens to 10th place	80
……..	……	……………	

88+1=	89	9 in unit place carry 8 tens to 10th place	80
89+1=	90	0 in unit place carry 9 tens to 10th place	90
90+1=	91	1 in unit place carry 9 tens to 10th place	90
....	......	....	90
98+1=	99	9 in unit place carry 9 tens to 10th place	90
99+1=	100	0 in unit place carry 1 hundred to 100th place	100
100+1=	101	1 in unit place carry 1 hundred to 100th place	100
....	....	....	......

2.5 Multiplication of Decimal Numbers:

Multiplication begins from unit place. The sign of multiplication is either . or × but in computer language it is also denoted by *

Example: Multiply 589 by 67

Solution:

$$
\begin{array}{lrl}
 & 5\,8\,9 & \\
 & \times\ 6\,7 & \\
 & \cdots\cdots\cdots & \\
7 \times 5\,8\,9 = & 4\,1\,2\,3 & \\
6 \times 5\,8\,9 = & +\ 3\,5\,3\,4\,0 & \text{[Why 0 comes in unit place?]} \\
 & \cdots\cdots\cdots & \\
 & 3\,9\,4\,6\,3 & \text{[Result of multiplication]}
\end{array}
$$

2.6 Techniques of Carry multiplication: (Description of above example)

a) First, multiply the 1st number by the digit of unit place of 2nd number beginning from the right. 7× 9 = 63. Put 3 and carry 6 to the next column. *The rule of carry is same with the addition*. Next, 7× 8 = 56 + 6 (carry) =62. Put 2 and carry 6 to the next column. Now, 7× 5 = 35 + 6 (carry) =41 and put it. First raw sub-result is 4123

b) Second, multiply the 1st number by the digit of 10th place of 2nd number. Here, the number in 10th place is 6 and its place value is 60. In multiplication this 0 of 60 comes to the unit place of 2nd-raw. Now, 6× 9 = 54. Put 4 and carry 5 to the next column. Next, 6× 8 = 48 + 5 (carry) =53. Put 3 and carry 5 to the next column. Now, 6× 5 = 30 + 5(carry) =35 and put it. The second raw sub-result is 35340

c) To get the final result add the sub-results of first and second row and it gives 39463.

2.7 Multiplication Rules of Decimal Numbers:

The multiplication 2576 × 632 is again conducted presenting the rules of multiplication in the following table 2.3.

Table 2.3: Rules of Multiplication

Place values ⇒	1000000th	100000th	10000th	1000th	100th	10th	Unit
1st number ⇒				2	5	7	6
(600 + 30 + 2) = 2nd number ⇒				×	6	3	2
(i) *2×2576* ⇒				*5*	*1*	*5*	*2*
(ii) *30×2576* ⇒			*7*	*7*	*2*	*8*	*0*
(iii) *600×2576* ⇒	*1*	*5*	*4*	*5*	*6*	*0*	*0*
Result=(i)+(ii)+(iii) ⇒	1	6	2	8	0	3	2

2.8 Subtraction of Decimal Numbers:

Subtraction is the inverse (reverse) operation of addition and it also begins from unit place.

Example: Subtract 567248 from 950034

Solution: 950034−567248

950034
−567248
....
382786 [Result] #

2.8.1 Subtraction Techniques (Description of the above example):

a) In the unit place (1st column from right) of the 1st row is 4 and it is less than 8 in the same column of the second row. Borrow 1 from 3 (the 2nd column and its place valueis 10) to unit place. After borrowing the new number in the 1st column is $4 + 10$ $(borrowing) = 14$ and performed the operation $14 - 8 = 6$ in the 1st column. Put 6 in the result section. After borrowing the remaining number in the 2nd column of 1st raw position is $3 - 1 = 2$

b) In the 1st row and 2nd column position the remaining number is 2 which is less than 4 of the same column. For borrowing it passes through two zeroes and borrows 1from 5, the 5th column and its place value is 10,000. This value distributed in 4th, 3rd and 2nd column as 9 Thousand, 9 Hundreds and 10 Tens respectively. Then new number in the 2nd column of 1st row is 2 **tens+10 tens=12** tens. Next, performed the operation 12−4=8 in the second column. Put 8 in the result section. The remaining number in the 5th column of 1st raw is $5 - 1 = 4$

c) In the 3rd column of 1st raw the remaining number is 9 (9 Hundreds) which is more than 2. Performed the operation 9−2=7. Put 7 in the result section.

d) In the 4th column of 1st raw the remaining number is 9 (9 Thousands) which is more than 7. Performed the operation $9 - 7 = 2$. Put 2 in the result section.

e) In the 5th column the remaining number is 4 which is less than 6 in the same column of the second row. Borrow 10 (10 ten thousands) from 6th place to 5th place. The new number in the 5th place is $4 + 10 = 14$. Performed the operation $14 - 6 = 8$ and put 8in the result section. The remaining number in 6th place of 1st row is $9 - 1 = 8$.

f) In the 6th place remaining number is 8. Performed the operation 8−5=3 and put 3 in the result section. Now, 382786 is the required result of subtraction.

2.8.2 Subtraction Rules:

a) In subtraction of decimal numbers 10 times of the place value is borrowed where necessary.

b) In case of borrowing which passes through one or more zeroes, it leaves 9 times of the place value in each place of zero.

Now, the rules of subtraction are illustrated in the following table 3.4 for the Subtraction 950034−567248

Table 3.4: Rules of Borrowing in Subtraction

Place values ⇒	100000th	10000th	1000th	100th	10th	Unit
Borrowing values and Remaining numbers ⇒	*8*	*10 ten thousands*				
		4	*9 thousands*	*9 hundreds*	*10 tens*	
					2	*10 unit*
1st Number⇒	9	5	0	0	3	4
−2nd Number⇒	5	6	7	2	4	8
Operation Performed ⇒	*8-5*	*10+4−6*	*9-7*	*9-2*	*10+2−4*	*10+4−8*
Result⇒	3	8	2	7	8	6

2.9 Division of Decimal Numbers:

Division is the inverse operation of multiplication. When a number (dividend) is divided by another smaller number (divisor) then it gives two numbers namely quotient and remainder. Division starts from upper most place value of the dividend. The sign of division is either ÷ or /

Example-1: Divide 357 by 4

Solution: Symbolically it can be written as $357 \div 4 = \frac{357}{4} = \frac{\textit{Dividend (Numerator)}}{\textit{Divison (Denominator)}}$

Division is performed as follows-

```
4 ]357[89
   32
```

```
  .............        [35−32=3]
   37                  [7 comes down from dividend]
   36
 ............
    1                  [37−36=1]
```

Here, Dividend= 357, Divisor= 4, Quotient= 89 and Remainder= 1 #

2.9.1 Division Techniques (Description of the example-1):

a) **Choose of the first dividend part:** Here, the divisor is single digit number 4. First take 3 the 1st digit of dividend from the left and it is less than 4. Now, take the next digit 5 of the dividend together with 3. Now it looks like 35 and it is the first dividend part.

b) **Choose of the greatest multiple of the divisor:** Multiply the divisor 4 by 1, 2, 3, 4, 5, 6, 7, 8, 9 and so on. In each multiplication observe the output number (multiple). Choose the greatest multiples which is less than or equal to 35. The required number is $4 \times 8 = 32$. Now, put 8 in the result (quotient) section and write down the number 32 below the number 35 and subtract it from 35. The subtraction result 3 is less than 4. This 3 is remainder of first part division.

c) **Choose of next digit/digits of dividend:** The next digit 7 of dividend comes down to the right of 3. These two numbers together constitute the new dividend part 37. Repeat the above technique (b) to choose the greatest multiple of 4. In this case the required multiple is $4 \times 9 = 36$. Put 9 in result section and write down the number 36 below 37 and subtract it from 37.

d) **Choose of Remainder:** The subtraction result is 1 and which is less than 4. There is no digit in the dividend to come down. So, 1 is remainder and 89 is quotient of the division.

Example-2: Divide 20137 by 25

Solution: 20137÷ 25

```
25 | 20137 | 805
   | 200   |
 ----------------
     137
     125
    -----
      12
```

Divisor = 25, Dividend= 20137, Quotient= 805, Remainder= 12 #

Description of the example-2:

a) **Choose of the first dividend part:** Here, the divisor is a two digits number 25. First take 20 the first two digits of the dividend from the left and it is less than 25. Now, take next digit 1 of the dividend together with 20. Now it looks like 201 and it is the first dividend part.

b) **Choose of the Greatest Multiple of the Divisor:** Multiply the divisor 25 by 1, 2, 3, 4, 5, and so on. In each multiplication observe the output number (multiple). Choose the greatest multiple which is less than or equal to 201. The required number is $25 \times 8 = 200$. Now, put 8 in the result (quotient) section and write down the number 200 below the number 201 and subtract it from 201. The subtraction result 1 is less than 25. This 1 is remainder of first part division.

c) **Choose of next digit/digits of the dividend:** The next digit 3 of dividend comes down to the right of 1. Now, the new number is 13 and it is less than 25. Put a 0 in the quotient and take the next digit 7 of the dividend in the right side of 13. Now, the new dividend is 137. Repeat the above technique (b) to choose the greatest multiple of 25. In this case the required multiple is $25 \times 5 = 125$. Put 5 in result section and write down the number 125 below 137 and subtract it from 137.

d) **Choose of Remainder:** The subtraction result is 12 and which is less than 25. There is no digit in the dividend to come down. So, 12 is remainder and 805 is quotient of the division.

2.10 Rules of Decimal Division:

a) Division is inverse operation of multiplication. The operation of multiplication begins from the digit in the unit place and moves towards the digits of upper place values. In case of division operation begins from the digit of uppermost place vale and move towards the digits of lower place values.

b) The first part of the dividend is so chosen that it is either equal or greater to the divisor and it play the role of dividend for first division.

c) The multiple of the divisor is so chosen that it is either equal or less than the first dividend taking under consideration.

d) The chosen multiple of divisor is subtracted from the first dividend part taking under consideration.

e) The number in the result of subtraction is less than the divisor and it is the remainder of first division.

f) The next number of the dividend (if available) comes down to the left of the remainder of first division. If the new number with this digit is either equal or greater than the divisor then the second time division is performed using the above rules.

g) If the new number is again less than the divisor and there is no digit in the dividend then one zero is put in the quotient part and the new number is the remainder. But, if there are available digits in the dividend then next digit comes down to the left of the remainder.

h) If this new number is again less than the divisor than another zero is put in the quotient and the number becomes remainder. But, if the new number is greater than the divisor, the next division is performed and so on.

2.11 Division Algorithm:

For any two integers n and, $d > 0$ $(n \geq d)$ there exist another two integers q and r such that $n = dq + r$, where $0 \leq r < d$ [Remember r is always positive]
Using capital letters the above algorithm can be narrated as

For any two integers N and, $(N \geq D)$ there exist another two integers Q and R such that $N = DQ + R$, where $0 \leq R < D$ and

$N = Numerator\ (Dividend)$
$D = Denominator(Divisor)$
$Q = Quotient$
$R = Remainder$

Note: $Dividend = Dvisor \times Quotient + Remainder.$

Example-3: Verify the division algorithm when 23 is divided by 7

Verification: $23 \div 7$

```
7 ] 2 3 [ 3
    2 1
.............
      2
```

Here, Divisor = 7, Dividend= 23, Quotient= 3, Remainder= 2

Now, $Dvisor \times Quotient + Remainder = 7 \times 3 + 2$
$= 23 =$ Dividend, verified #

2.12 Division of a negative number by a positive number:

When a negative number is divided by a positive number then it should be remember that the quotient may be negative, but the remainder is always positive.

Example-4: Verify division algorithm when -29 is divided by 8

Verification: $-29 \div 8$

8]−2 9 [−4
  − 3 2
  ....
     3

$[-29-(-32) = -29+32 = 3]$

[remainder is always positive]

Here, Divisor = 8, Dividend= -29, Quotient= -4, Remainder= 3

Now, $Dvisor \times Quotient + Remainder = 8 \times (-4) + 3$

$= -32 + 3 = -29 =$ Dividend, verified #

Note: In general $\frac{-a}{b} = -(\frac{a}{b})$

2.13 Divisible Pair of Numbers:

When a number n is divided by a number d and leaves remainder 0, then it is said that n is divisible by d. Sometimes, it is denoted by $d|n$ and read as"d $divides$ n". Here, (n, d) is divisible pair. e.g. 3|6 and (6, 3) is divisible pair.

Example: 28 is divisible by 7

Verification:

$28 \div 7$

7] 2 8 [4
  2 8
  ....
   0

When 28 is divided by 7 then it leaves remainder 0. Hence, 28 is divisible by 7.

2.14 Test of divisibility of a number by prime number.

a) **Divisible by 2:** All the even numbers are divisible by 2. A number is even number if the digit in the end is either of the digits 0, 2, 4, 6, 8.
Examples: 130, 932, 574, 536, 158 etc. are divisible by 2

b) **Divisible by 3:** When the sum of the digits of a number is divisible by 3, then it is divisible by 3.
Examples:
474: 4+7+4=15 and 15 is divisible by 3. Therefore, 474 is divisible by 3
1323: 1+3+2+3=9 an 9 is divisible by 3. Therefore, 1323 is divisible by 3

c) **Divisible by 5:** The numbers with end digit either 0 or 5 is divisible by 5
Examples: 10, 15, 20, 750, 3715, etc. are divisible by 5

d) **Divisible by 7:** A number is divisible by 7 if the twice of the end digit differs with the remaining part of the number by either 0 or multiple of 7

Example: 238 is divided by 7

$23 - (8 \times 2) = 23 - 16 = 7$, which is multiple of 7(divisible by 7)

Example: 126 is divided by 7

$12 - (6 \times 2) = 12 - 12 = 0$

Example: similarly, 84, 238, 546 etc. are divisible by 7

e) **Divisible by 11:** A number is divisible by 11 if the sum of digits at odd and even places is equal or differ by a number multiple of 11.

Example: 235279 is divisible by 11

Sum of the digits in odd places=2+5+7=14

Sum of the digits in even places=3+2+9=14

Example: 121 is divisible by 11

Sum of the digits in odd places=1+1=2

Sum of the digits in even places=2

Example: 72831 is divisible by 11

Sum of the digits in odd places=7+8+1=16

Sum of the digits in even places=2+3=5

Difference between the sum= 16−5=11, which is multiple of 11

f) **Divisible by 13:** A number is divisible by 13 if 9 times of the last digit differs from the remaining part of the number by either 0 or multiple of 13

Example: 156 is divisible by 13

$15 - (6 \times 9) = 15 - 54 = -39$, difference is 39 and it is multiple of 13

Example: 871 is divisible by 13

$87 - (1 \times 9) = 87 - 9 = 78$, and 78 is multiple of 13(divisible by 13)

Repeat the process for 78:

$7 - (8 \times 9) = 7 - 72 = -65,\ 65\ is\ multiple\ of\ 13$

Repeat the process for 65:

$6 - (5 \times 9) = 6 - 45 = -39,\ 39\ is\ multiple\ of\ 13$

Example: 637 is divisible by 13

$63 - (7 \times 9) = 63 - 63 = 0$

g) **Divisible by 17:** A number is divisible by 17 if 5 times of the last digit differs from the remaining part of the number by either 0 or multiple of 17

Example: 153 is divided by 17

$$15 - (3 \times 5) = 15 - 15 = 0$$

Example: 697 is divisible by 17

$$69 - (7 \times 5) = 69 - 35 = 34, \text{ multiple of 17 (divisible by 17)}$$

h) **Divisible by 19:** A number is divisible by 19 if 2 times of the last digit when added to the remaining part of the number then it becomes multiple of 19.

Example: 1083 is divisible by19

$$108 + (3 \times 2) = 108 + 6 = 114, \text{ multiple of 19}$$

Repeat the process for 114:

$$11 + (4 \times 2) = 11 + 8 = 19, \text{ multiple of 19}$$

i) **Divisible by 23:** A number is divisible by 23 if 7 times of the last digit when added to the remaining part of the number then it becomes multiple of 23.

Example: 621 is divisible by 23

$$62 + (1 \times 7) = 62 + 7 = 69, \text{ multiple of 23}$$

Example: 943 is divisible by 23

$$94 + (3 \times 7) = 94 + 21 = 115, \text{ multiple of 23}$$

Repeat the process for 115:

$$11 + (5 \times 7) = 11 + 35 = 46, \text{ multiple of 2}$$

2.15 Test of divisibility of a number by composite number.

A composite number is always divisible by the product of its any number of factors.

$4 = 2 \times 2$ is divisible by 2, 4

$6 = 2 \times 3$ is divisible by 2, 3 and 6

$8 = 2 \times 2 \times 2$ is divisible by 2, 4 and 8

$10 = 2 \times 5$ is divisible by 2, 5 and 10

$12 = 2 \times 2 \times 3$ is divisible by 2, 3, 4, 6 and 12

$14 = 2 \times 7$ is divisible by 2 and 7

etc.

2.16 Some special rules of divisibility by composite numbers:

a) **Divisible by 4:** A number is divisible by 4 if the number formed by the last two digits of the number is divisible by 4

Example: 5736 is divisible by 4. Because, 36 is divisible by 4

b) **Divisible by 6:** A number is divisible by 6 if it is divisible by 2 and 3

Example: 3564 is divisible by 6

3564 is even number and it is divisible by 2

$3 + 5 + 6 + 4 = 18$ and it is divisible by 3

c) **Divisible by 8:** A number is divisible by 8 if the number formed by the last three digits of the number is divisible by 8.

Again, if in a number the last three digits are zero, then the number is divisible by 8

Example: 3176 is divisible by 8.

$176 = 8 \times 22$, divisible by 8

Example: 371000 is divisible by 8 [last three digits are zero]

d) **Divisible by 10:** A number ending with zero is divisible by 10

Examples: 20, 30, 170, 2300, etc. are divisible by 10

2.17 Properties of Divisibility:

(i) $a | \pm a$ e.g. $3|3,\ 3|-3$

(ii) $a|\,b$ and $b|\,c \Rightarrow a|\,c$ e.g. $7|14$ *and* $14|28 \Rightarrow 7|28$

(iii) $a|\,b$ and $a|\,c \Rightarrow a|\,b \pm c$ e.g. $5|10$ *and* $5|15 \Rightarrow 5|10 \pm 15$

(iv) $a|\,b$ and $c|\,d \Rightarrow ac|\,bd$ e.g. $2|4$ *and* $3|9 \Rightarrow 2 \times 3|4 \times 9$

(v) $ac|\,bc \Rightarrow a|\,b$ e.g. $6|36 \Rightarrow 2 \times 3|12 \times 3 \Rightarrow 2|12$

Chapter-3

Real Numbers

3.1 The Real Numbers:

The real numbers are continuous numbering system. There is no gap between two real numbers. A real number represents distance along the real axis with respect to the point zero (o). The real numbers system is divided into 5 categories (subsets) namely natural numbers, integers (whole numbers), rational numbers and irrational numbers. Each of the categories is proper subset of the real numbers system.

3.2 The Natural Numbers:

The natural numbers are counting numbers. Therefore, zero (0) and negative numbers are excluded from the natural numbers system. The set of natural numbers is infinite set and it is denoted by N.

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots..\}$$

Symbolically, the set of natural numbers = $\{ \mid x \in N \}$

3.2.1 The Whole numbers:

All the natural numbers along with 0(zero) is called whole numbers. The set of whole numbers is infinite and it is denoted by W

$$W = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots..\}$$

3.2.2 The Odd numbers:

A natural number which is not divisible by 2 is called odd number. An odd number is denoted by either $(2n - 1)$ or $(2n + 1)$, where n$\in N$. The set of odd numbers can be represented by D [the symbol is not recognized].

$$D = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots \ldots\}$$
$$= \{2n - 1 \mid n \in N\}$$

3.2.3 The Even numbers:

A natural number which is divisible by 2 is called even number. An even number is denoted by $2n$, where n$\in N$. The set of even numbers can be represented by E [the symbol is not recognized].

$$E = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots \ldots\}$$
$$= \{2n \mid n \in N\}$$

3.3 The Prime Numbers:

A natural number is called a prime number if it has only two factors 1 and the number itself. That is a prime number is not divisible by any other number accept 1 and itself. The number 2 is the smallest and only even prime. Examples of prime numbers are

$$2 = 2 \times 1$$
$$3 = 3 \times 1$$
$$5 = 5 \times 1$$
$$7 = 7 \times 1 \text{ ....etc}$$

The set of prime numbers is infinite set and it is denoted by P.

$$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots\ldots\ldots\}$$

Table 3.1: Prime numbers between 1 and 100

	11		**31**	**41**		**61**	**71**		
2									
3	**13**	**23**		**43**	**53**		**73**	**83**	
5									
7	**17**		**37**	**47**		**67**			**97**
	19	**29**			**59**		**79**	**89**	

3.3.1 Properties of prime numbers:

Every prime number greater than 3 can be expressed in the form either $(6n + 1)$ or $(6n - 1)$, where n is a natural number.

Verification of the property: Consider the prime numbers 79, 89 and 104729.

$79 = 6 \times 13 + 1, \quad 13 \in N$

$89 = 6 \times 15 - 1, \quad 15 \in N$

$104729 = 6 \times 17455 - 1, \;\; 17455 \in N.$ verified #

Note: The converse of the above property is not always true. If a number is of the form either $(6n + 1)$ or $(6n - 1)$ then it may not be a prime.

Counter Example: $143 = 6 \times 24 - 1$, which is of the form $6n + 1$

But, 143 is not a prime. Because, 143 is divisible by 11

3.3.2 Primality Test:

To test a given number p is prime or not

Step-1: Verify -is the number p expressible in the from either $(6n+1)$ or $(6n-1)$? If answer is 'no' then it is not prime. If answer is 'yes' then further verification is required.

Step-2: Find a least natural number x such that $x > \sqrt{p}$

Next, find all the prime numbers less than or equal to x

If p is not divisible by any one of these primes, then p is a prime number.

Example-1: Is the number 167 is prime?

Solution:

Step-1: $167 = 6\times 28 - 1$, where $28 \in N$

Step-2: $13 \times 13 = 169$. That is $13 = \sqrt{169}$

Clearly $13 > \sqrt{167}$

Now, prime numbers less than or equal to 13 are-

2, 3, 5, 7, 11, 13 and neither one of these prime divides 167

Hence, 167 is a prime number. #

3.3.3 Co-prime Numbers:

Co-prime numbers are relatively prime or mutually prime numbers. Two numbers are said to be co-prime if their common factor is 1. In other words two natural numbers x, y are said to be co-prime if the greatest common factor (GCF) between them is 1.

i.e. $\text{GCF}(x, y) = 1$

Examples-2: The pairs $(2,3)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, $(5, 6)$, $(4, 9)$ etc. are co-prime numbers.
Example-3: $\{2, 7, 9, 15\}$ is a set of co-prime numbers.

3.4 The Composite numbers:

A non-prime and non-unit natural number is called composite number. A composite number must have at least one factor other than 1 and the number itself. The least composite number is 4.

Examples of composite numbers are $4 = 2 \times 2$, $6 = 2 \times 3$, etc.

Note: The number 1 is neither prime nor composite.

3.5 The Set of Integers:

The set of negative integers, the set of positive integers and 0 constitute the set of integers. It is infinite set and denoted by Z or I

$$Z = \{\dots\dots.-5,-4,-3,-2,-1\} \cup \{0\} \cup \{1,2,3,4,5,\dots\dots\}$$

$$Z = \{\ldots\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\ldots\}$$

3.5.1 The Number line of Z: There is 1 unit gap between two integers.

−∞ ←——|—|—|—|—|—|—|—|—|—|—|—|—|——→ +∞

.................-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6..........

3.6 The Set of Rational Numbers:

The set of rational numbers is infinite set and it is denoted by Q. It is defined as

$$Q = \{ \frac{a}{b} \mid b \neq 0 \text{ and } a, b \in Z\}$$

where the denominator $b \neq 0$ and a, b, are integers.
The examples of rational numbers are

$$0 = \frac{0}{1}, \quad 1 = \frac{1}{1}, \quad 2 = \frac{2}{1}, \quad 3 = \frac{3}{1}, \quad 4 = \frac{4}{1} \ \ldots\ldots\ldots\ \text{etc.}$$

....

$$-1 = \frac{-1}{1}, \quad -2 = \frac{-2}{1}, \quad -3 = \frac{-3}{1}, \quad -4 = \frac{-4}{1} \ \ldots\ldots\ldots\ \text{etc.}$$

....

$$0.1 = \frac{1}{10}, \quad 0.2 = \frac{2}{10}, \quad 0.3 = \frac{3}{10}, \quad 0.4 = \frac{4}{10} \ \ldots\ldots\ldots\text{etc.}$$

$$0.01 = \frac{1}{100}, \quad 0.02 = \frac{2}{100}, \quad 0.03 = \frac{3}{100}, \quad 0.04 = \frac{4}{100} \ \ldots\ldots\ldots\text{etc.}$$

....

3.7 The Set of Irrational numbers:

The numbers which are not rational is called irrational numbers. An irrational number cannot be expressed in the form $\frac{a}{b}$,where $b \neq 0$. There is no standard notation to represent the set of irrational numbers. But sometimes it is represented as the complement of Q, the set of rational numbers. The set of irrational numbers is infinite set.

Set of irrational numbers = $Q^c = R \setminus Q$, where R is the set of real numbers.

$$= R - Q$$

After the decimal point each irrational number has infinite numbers of digits with regularly ***non-repeating*** *digits or group of digits.*

Example-4: The irrational numbers are Pi (π), Euler's number (e), Golden ratio (φ *or* $\emptyset$), $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{13}$, $\sqrt{15}$, ... etc.

$$\pi = \frac{Circumference\ of\ circle}{diameter}$$ [π is first introduced by William Jones in 1706]

In fact π cannot be expressed as quotient of two integers. Also, it is not exactly equal to $\frac{22}{7}$, it is approximately equal to $\frac{22}{7}$

$$\frac{22}{7} \cong \pi = 3.14159265358979323 8 \ldots\ldots\ldots \cong 3.14159$$

Similarly,

$$e = 2.718281828459045 \ldots\ldots\ldots\ldots \cong 2.71828$$

$$\varphi = 1.618033988749 \ldots\ldots\ldots\ldots \cong 1.61803$$

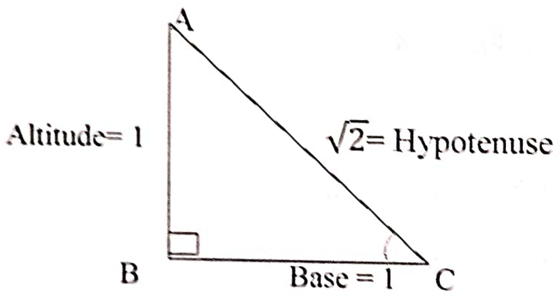
$$\sqrt{2} = 1.4142135623730950 \ldots\ldots\ldots \cong 1.414$$

$$\sqrt{3} = 1.7320508075688772935 \ldots\ldots \cong 1.732$$

Note: Actual Value of $\frac{22}{7} = 3.142857142857 = 3.\overline{142857}$

3.7.1 Geometrical Representation of Irrational Numbers:

(i) Let us consider the isosceles right angled triangle ABC with Base=altitude=1. Then its hypotenuse is equal to $\sqrt{2}$



[Fig-7]

Here, $m < B = 90°$, AB=BC=1. Hence, by Pythagoras theorem in right angled triangle

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow AC = \sqrt{AB^2 + BC^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

(ii) Let us consider the equilateral triangle ABC with length of each side 2 units. Then the altitude of the triangle is equal to $\sqrt{3}$

(iii) Let us consider the right angled triangle with base 2 units and altitude 1 unit. Then its hypotenuse is equal to $\sqrt{5}$

(iv) Let us consider the right angled triangle with base 3 unit and altitude 1 unit. Then its hypotenuse is equal to $\sqrt{10}$

Theorem: If a be a positive integer and a prime number p divides a^2 then p divides a.

[With the help of this theorem we shall prove that $\sqrt{2}$ is irrational]

Example-5: Prove that $\sqrt{2}$ is irrational.

Proof: If possible let $\sqrt{2}$ be rational. Then it can be expressed as-

$\sqrt{2} = \frac{a}{b}$, where a, b are co-prime

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow 2b^2 = a^2$ ……… (1)

$\Rightarrow$ 2 divides a^2 and 2 is prime

$\Rightarrow$ 2 divides a [by the above theorem]

$\Rightarrow$ 2 is a factor of a ………….(2)

$\Rightarrow a = 2 \times c$, where c is an positive integer

Putting this value in (1) it gives

$\Rightarrow 2b^2 = (2 \times c)^2$

$\Rightarrow 2b^2 = 4c^2$

$\Rightarrow b^2 = 2c^2$

$\Rightarrow$ 2 divides b^2 and 2 is prime

$\Rightarrow$ 2 divides [by the above theorem]

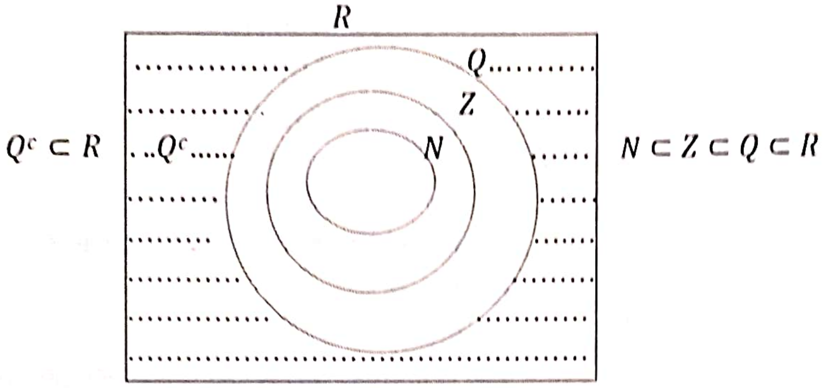
$\Rightarrow$ 2 is a factor of b ………………….. (3)

From (2) and (3) it is clear that 2 is a common factor of a *and* b, which contradicts to the fact that a, b are co-prime. Hence, our assumption that $\sqrt{2}$ is a rational was wrong. Therefore, $\sqrt{2}$ is an irrational number. Proved#

3.8 The Set of Real Numbers:

The set of real numbers is infinite set and it is denoted by R.The set of rational and irrational numbers are including in the set of real numbers.
$R = Q \cup Q^c$, where $Q = The\ set\ of\ rationals$ and $Q^c = The\ set\ of\ irrationals$

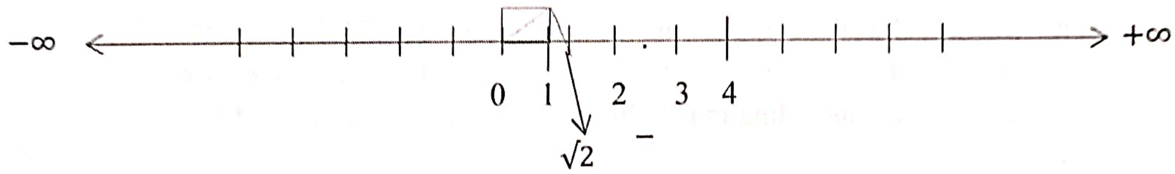
The doted area in the following Venn diagram-8 represents the set of irrational numbers.



(Venn diagram of Real number system)
(Fig-8)

3.8.1 The Real Axis (Line):

The real numbers can be represented continuously by the points in a directed line. This straight line is called real axis. Between two real numbers there is no gap.

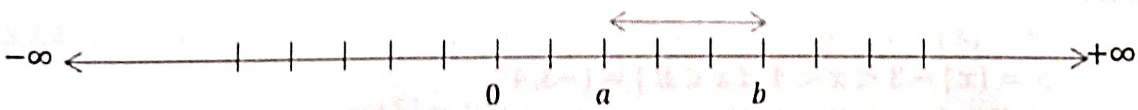


(Fig-9)

For every real number there is a point on the real axis and for every point on the real axis there is a fixed real number. There is one-one correspondence between real numbers and points on the real axis.
Let us consider a small part of the real axis between 0 and 1
Nearer point to the left of 1 are 0.1, 0.01, 0.001, 0.0001, 0.00001 .... ∞
That is there are infinite numbers of points between 0 and 1. It is impossible to display all the real numbers between 0 and 1. To overcome this situation the idea of interval is introduced in mathematics.

3.8.2 Open Interval and Closed Interval:

Let us consider two points a and b on the real axis such that $a < b$



Open Interval is defined as $(a, b) = \{x \mid a < x < b \ and \ x \in \boldsymbol{R} \}$
= Set of all real numbers between a and b
Where, the end points a and b are excluding

Closed interval is defined as $[a, b] = \{x \mid a \leq x \leq b \ and \ x \in \boldsymbol{R} \}$
= The set of all real numbers between a and b
together with the end points a and b

Open-closed interval: $(a, b] = \{x \mid a < x \leq b \ and \ x \in \boldsymbol{R} \}$
= The set of all real numbers between a and b
together with the end point b

Closed-open interval: $[a, b) = \{x \mid a \leq x < b \ and \ x \in \boldsymbol{R} \}$
= The set of all real numbers between a and b
together with the end point a

Note:

1. (a, b) *and* $[a, b]$ are infinite subsets of real numbers
   That is $(a, b) \subset R$ and $[a, b] \subset R$
2. The elements a, b are excluding in the open interval (a, b). i.e. $a, b \notin (a, b)$
3. The elements a, b are including in the closed interval $[a, b]$. i.e. $a, b \in [a, b]$

Examples-6:

1. $\{x \mid 2 \leq x \leq 5 \ and \ x \in \ \} = [2, 5]$, which is infinite set
   But, $\{x \mid 2 \leq x \leq 5 \ and \ x \in \boldsymbol{Z} \} = \{2, 3, 4, 5 \}$, which is finite set

2. $\{x \mid 3 < x < 8 \ and \ x \in \ \} = (3, 8)$, which is infinite set
   But, $\{x \mid 3 < x < 8 \ and \ x \in \boldsymbol{Z} \} = \{4, 5, 6, 7\}$, which is finite set

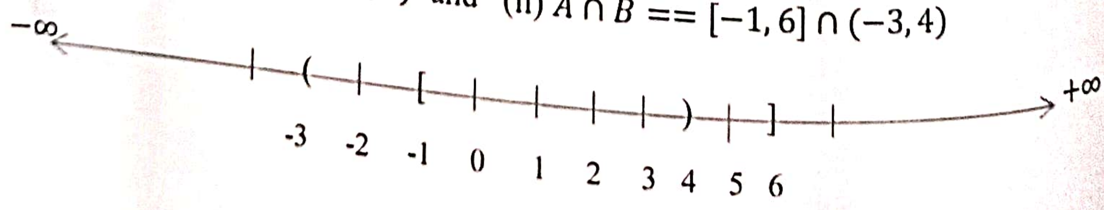
3. $\{x \mid -2 \leq x < 7 \} = [-2, 7)$, when the set is not mentioned, then take it as R

4. $\{x \mid 0 < x \leq 4 \} = (0, 4]$, when the set is not mentioned, then take it as R

Example-7: Given $A = \{x \mid -1 \leq x \leq 6 \ and \ x \in \boldsymbol{R} \ \}$ and
$B = \{x \mid -3 < x < 4 \ and \ x \in \boldsymbol{R}\}$
Find (i) $A \cup B$ and (ii) $A \cap B$

Solution:

$A = \{x \mid -1 \leq x \leq 6 \ and \ x \in \ \} = [-1, 6]$ and
$B = \{x \mid -3 < x < 4 \ d \ x \in \boldsymbol{R} \} = (-3, 4)$
(i) $A \cup B = [-1, 6] \cup (-3, 4)$ and (ii) $A \cap B == [-1, 6] \cap (-3, 4)$



(i) $A \cup B = [-1,6] \cup (-3,4) = (-3,6]$
$= \{x \mid -3 < x \leq 6\}$

(ii) $A \cap B == [-1,6] \cap (-3,4) = [-1,4)$
$= \{x \mid -1 \leq x < 4\}$ #

Example-7: Given $A = \{x \mid -1 \leq x \leq 6 \; and \; x \in \mathbf{Z}\}$ and
$B = \{x \mid -3 < x < 4 \; and \; x \in \mathbf{Z}\}$
Find (i) $A \cup B$ and (ii) $A \cap B$

Solution: (in this case the given set is Z)
$A = \{x \mid -1 \leq x \leq 6 \; and \; x \in \mathbf{Z}\} = \{-1, 0, 1, 2, 3, 4, 5, 6\}$ and
$B = \{x \mid -3 < x < 4 \; and \; x \in \mathbf{Z}\} = \{-2, -1, 0, 1, 2, 3\}$

(i) $A \cup B = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$

(ii) $A \cap B = \{-1, 0, 1, 2, 3\}$ #

3.8.3 Modulus (Absolute value) of Real Numbers:

Let, x be any real number. Then modulus of x is denoted by $|x|$ and is defined as
$|x| = x, \quad if \; x \geq 0$ [x is positive or equal to zero]
$= -x, \quad if \; x < 0$ [x is negative]

Note: $|x|$ is always positive

Example-8: If $x = 3$, then $3 > 0$ and $|3| = 3$, a positive number.
If $x = -3$, then $-3 < 0$ and $|-3| = -(-3) = 3$, a positive number.
That is $|3| = |-3| = 3$, modulus represent only a numerical value.

Example-9: The real number nearer to 1 is
(A)0.9 (B) 0.99 (C) 1.1 (D) 1.01 (E) None of the given.
Answer: (E) **Note:** There are infinite numbers of points nearer to 1

Example-10: Between two adjacent numbers there is no gap if they are
(A) Natural numbers
(B) Rational numbers
(C) Integers
(D) Real numbers **Answer:** (D)

3.8.4 Sign Rules of Real Numbers:

<u>**Additive and Subtractive Sign Rules**</u>

Let, a, b be any positive real numbers such that $a < b$, then

$$a + a = 2a$$

$$(-a) + (-a) = -2a$$

$$a - a = 0$$

$$-a + a = 0$$

$$a - b = (-)\ a\ negative\ number \quad [a < b]$$
$$b - a = (+)\ a\ positive\ number \quad [b > a]$$

Multiplicative Sign Rules:

For any positive real numbers $a\ and\ b$

$$a \times b = ab$$

$$(-a) \times (-b) = ab$$

$$(-a) \times b = -ab$$

$$a \times (-b) = -ab$$

3.8.5 Properties of real numbers:

I. For any two elements $a, b \in R$, one and only one of the following relations is true

$a < b,\ \ a = b,\ \ a > b$

II. $a > b\ and\ b > c \Rightarrow a > c$

III. $a > b \Rightarrow a + c > b + c$

IV. $a > b\ and\ c > 0 \Rightarrow ac > bc$

V. $a > b\ and\ c < 0 \Rightarrow ac < bc$

Chapter-4

Binary Numbers

4.1 Binary Numerals:

In binary numeral system numbers are written using only two digits 0 (zero) and 1 (one) in the base-2 numeral system. Each digit is referred to as a bit. It is very simple mathematical language and widely used as logic gate in electronic circuit, computer programming and computer-based devices.

Table 4.1

Decimal:	0	1	2	3	4	5	6	7	8	9	10	11	12
Binary:	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100

The binary numbers are often subscripted the base and written as $(1011)_2$ which is equal to 11 in decimal system. A binary number is read digit-by-digit (bit-by-bit). For example binary number 1011 is read as *one zero one one.*

4.2 Addition Properties of Binary Numbers:

To write a binary number we need to know the following addition properties of binary numbers.

$$0+0=0$$
$$1+0=1$$
$$0+1=1$$
$$1+1=10 \text{ (Result 0 and carry 1 to the next left-column)}$$
$$1+1+1=11 \text{ (Result 1 and carry 1 to the next left-column)}$$
$$1+1+1+1=100 \text{ (Result 0 and carry 10 to the next two-columns)}$$

First start from the bit 0 (zero). Then add 1 in every step to have the successive binary numbers. In each column of the following table 4.2 the bit 1 is successively added starting from the bit 0 to get the successive numbers in base-2 binary system.

Table 4.2: The successive binary numbers in the form of addition

First	*0* *+1*	*1* *+1*	*10* *+1*	*11* *+1*	*100* *+1*	*101* *+1*	*110* *+1*	*111* *+1*	*1000* *+1*	*1001* *+1*	*1010* *+1*	*1011* *+1*	*1100* *+1*
0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101

Example-1:

(i)
```
 110110
+11111
.....................
1010101
```

(ii)
```
 1000101
+101101
........................
1110010
```

(iii)
```
 11111
+ 111
...........
100110
```

Answers: **(i)** (1010101), **(ii)** $(1110010)_2$, **(iii)** $(100110)_2$

4.3 Big Carry Addition:

Let us consider the following addition to illustrate the technique and rules of big ca binary addition.

```
 11111
  1111
   111
......+...................
110101      (Result)=(110101)2 #
```

4.3.1 Techniques of Big Carry Addition: (Description of the above example)

1. Addition in 1st column from the right gives 11. Now result is 1 and carry is 1 to the n column.
2. Addition in 2nd column gives $1+1+1+1(cary)=100$. Now result is 0 and ca is 10 to the next two columns.
3. Addition in 3rd column gives $1+1+1+0(carry)=11$. Now result is 1 and carr is 1 to the next column.
4. Addition in 4th column gives $1+1+1(carry\ of\ 2nd\ column)+1(cary\ of\ 3rd\ column)=100$. Now result is 0 and carry is 10 to the next two columns.
5. Addition in 5th column gives $1+0(carry)=1$ and result is 1
6. Addition of 6th column gives 1(carry)=1 and result is 1
7. The result of addition is $(110101)_2$

4.3.2 Rules of Big Carry addition:

The rules of binary addition are illustrated step-by-step in the following tal In the table the 1st, 2nd and 3rd binary numbers for addition are placed in the mid rows. The upper four rows are for counting carry bits. Actual value of column addit is also showed for each column in the table 4.3. The bits of the last row give the ansv of the addition.

Table 4.3: Big carry Binary Addition

No. of Column	6th	5th	4th	3rd	2nd	1st		Decimal Representation
Carry from 4th column	*1*	*0*					Carry Numbers	
Carry from 3rd column			*1*					
Carry from 2nd column			*1*	*0*				
Carry from 1st column					*1*			
Numbers to be added		1	1	1	1	1		31
			1	1	1	1		15
				1	1	1		7
Columns Addition	*1*	*1*	*100*	*11*	*100*	*11*		
Resulting bits	1	1	0	1	0	1		53

4.4 Place Value of Binary digit (Expression from Binary to decimal Number):

In the binary system the value of each bit (digit) in a number is represented by the power of 2. The successive place values of binary bits in a numbers are presented in the following table 4.4.

Table 4.4: Place value of binary digit

6th	5th	4th	3rd	2nd	1st	Binary point	1st	2nd	3rd	4th
$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$	.	$2^{-1} = 0.5$	$2^{-2} = 0.25$	$2^{-3} = 0.125$	$2^{-4} = 0.0625$

Note: The value of a binary number is the Sum of the Products $(\boldsymbol{digit \times place\ value})$

$$= \sum \boldsymbol{digit \times place\ value}$$

Solved Example-2:

(i) $(1011)_2 = 1\times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3$ [From right to left]

$= 1\times 1 + 1 \times 2 + 0 \times 4 + 1 \times 8 \quad = 11$ (eleven)

(ii) $(11010)_2 = 0\times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 = 26$ (twenty-six)

Now, let us consider the following binary fraction-

(iii) $(0.1101)_2 = 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4}$ [From left to right]

$= 1 \times 0.5 + 1 \times 0.25 + 0 \times 0.125 + 1 \times 0.0625$

$= 0.5 + 0.25 + 0 + 0.0625$

$= (0.8125)_{10}$, a decimal fraction

4.5 Conversion of decimal Number to Binary Number:

Let us consider the decimal number 37. Now, successively divide 36 by 2 until the value becomes 0 and in each division count the remainder. Next, arrange the remainder beginning *from bottom to top* in horizontal line to have the corresponding binary number.

Example-3: Convert the decimal number 37 to binary number.

Solution: Let us divide the number 37 successively and keep count of remainders

$37 \div 2 = 18$ plus remainder 1

$18 \div 2 = 9$ plus remainder 0

$9 \div 2 = 4$ plus remainder 1

$4 \div 2 = 2$ plus remainder 0

$2 \div 2 = 1$ plus remainder 0

$1 \div 2 = 0$ plus remainder 1

Hence, binary equivalent of $(37)_{10}$ is $(100101)_2$

Therefore, $37 = (100101)_2$

Example-4: Convert the number 43 to binary form

Solution: $43 \div 2 = 21$ plus remainder 1

$21 \div 2 = 10$ plus remainder 1

$10 \div 2 = 5$ plus remainder 0

$5 \div 2 = 2$ plus remainder 1

$2 \div 2 = 1$ plus remainder 0

$1 \div 2 = 0$ plus remainder 1

Next, arrange the remainders beginning from bottom to top in horizontal line

Therefore, $43 = (101011)_2$

4.6 Conversion of decimal Fraction to Binary fraction:

The binary fraction can be found applying successive multiplications to a decimal fraction by 2 and keeping the counts of integral parts in each multiplication. The successive multiplication terminates when decimal part becomes 0 (zero). The entire process of conversion is described in the following table 4.5.

Table 4.5 Conversion from decimal fraction to binary fraction

Decimal Fractions	Successive multiplication of decimal fraction by 2	Carry Integral Parts	Binary Fraction
$\frac{1}{2}$=0.5	0.5 × 2 = 1.0	1	0.1
$\frac{1}{3}$=0.3333...	0.3333...× 2 = 0.6666 0.6666...× 2 = 1.3332 0.3332....× 2= 0.6664 0.6664....× 2 = 1.3328 0.3328...× 2 =0.6656 0.6656..× 2 =1.3312	0 1 0 1 0 1	0.010101... (arrange the integral parts *beginning from top to bottom* in horizontal line)
$\frac{1}{4}$=0.25	0.25 × 2 = 0.5 0.5 × 2 = 1.0 (decimal part is 0, multiplication terminates)	0 1	0.01
$\frac{1}{5}$=0.2	0.2 × 2 = 0.4 0.4 × 2 = 0.8 0.8 × 2 = 1.6 0.6 × 2 = 1.2 0.2 × 2 = 0.4 0.4 × 2 = 0.8 0.8 × 2 = 1.6 0.6 × 2 = 1.2	0 0 1 1 0 0 1 1	0.00110011....
$\frac{1}{6}$=0.1666...	Proceeding as in above		0.0010101.....
$\frac{1}{7}$=0.142857...			0.001001.....
$\frac{1}{8}$=0.125			0.001
$\frac{1}{9}$=0.1111...			0.000111000111....
$\frac{1}{10}$=0.1			0.000110011.....

4.7 Multiplication in Binary System:

Multiplication rules of binary numbers are same with that of decimal system. But addition follows the binary rules. There are only two digits 0 and 1 in binary system.

Table 4.6: Binary multiplication

1 × 0 =	0
0× 1 =	0
1× 1 =	1

Now, let us performed the multiplication of two binary numbers $(11110)_2$ by $(101)_2$ using the following table 4.7

Table 4.7: Multiplication table of $(11110)_2 \times (101)_2$

Numbers to be multiplied				1	1	1	0	1	
					×	1	0	1	
11110 × 1=				1	1	1	0	1	Binary addition of three rows
11110 × 0(place value 00)=			0	0	0	0	0	0	
11110 × 1 (place value 100)=		1	1	1	0	1	0	0	
Result=	1	0	0	1	0	0	0	1	

That is $(11101)_2 \times (101)_2 = (10010001)_2$ is the required answer.

Example-5: Evaluate the following multiplications.

```
          1 1 0 1 1 1
           × 1 0 1 1
        ........................
           1 1 0 1 1 1        [result multiplied by 1]
         1 1 0 1 1 1 0        [result multiplied by 1(place value 10)]
 +     0 0 0 0 0 0 0 0        [result multiplied by 0 (place value 000)]
     1 1 0 1 1 1 0 0 0        [result multiplied by 1(place value 1000)]
   ..............................
   1 0 0 1 0 1 1 1 0 1
```

The required result is $(1001011101)_2$ #

4.8 Subtraction in Binary System:

The subtraction in binary system follows the same rules as with that of decimal system But carry numbers are different due to base-2 system. In decimal system the borrowed numbers are group of 10_{10} (borrowed 10 times from one place to the next lower place). In binary system the borrowed numbers are group of 2_{10} which is equivalent to 10_2.

Table 4.8: Basic rules of binary subtraction

0 – 0 =	0
1 – 0 =	1
0 – 1 =	1 (borrow 1 from the next more significant bit
1 – 1 =	0
10 – 1 =	1

Example-6: To subtract $(00111)_2$ from $(11100)_2$
i.e $(11100)_2 - (00111)_2$

Solution:

For the better understanding of the first readers subtraction is performed using the following table-4.9. The bits of the two numbers are arranged in the middle rows of the table. The upper two rows are for counting carry bits. The actual operation performed is also showed for each column in the table 4.9. The bits of the last row give the answer of the subtraction.

Table 4.9: Rules of borrowed binary subtraction (Taking borrowed 10_2)

5^{th}	4^{th}	3^{rd}	2^{nd}	1^{st}	Column Count	
	0	*10*			*When Borrowed 1 from 4^{th} to 3^{rd} column then the remaining bit/bits in the 3^{rd} and 4^{th} column are 10, 0 respectively*	
		0	*1*	*10*	*When Borrowed 1 from 3^{rd} to 1^{st} column then the remaining bit/bits in 1^{st}, 2^{nd} and 3^{rd} column are 10, 1, 0 respectively*	
1	**1**	**1**	**0**	**0**	1^{st} Number	Subtraction
0	**0**	**1**	**1**	**1**	-2^{nd} number	
$1-0$	$0-0$	$10-1$	$1-1$	$10-1$	*Operation performed*	
1	**0**	**1**	**0**	**1**	Result of subtraction	

The answer of subtraction is $(10101)_2$

Note:

1. To subtract the column bit of 2^{nd} number from the borrowed bit/bits of the same column.
2. When 1 is borrowed from 3^{rd} column to 1^{st} column, then actual value of browed is 100. In the 2^{nd} column place value of 1 is 10. It gives 0+10+10= 100

4.8.1 Alternative method of binary subtraction (taking borrowed group of 2_{10}):

Binary subtraction can also be done by borrowing 2. The details techniques of borrowing are illustrated in the following table 4.10. for the subtraction of the binary number 111 from 11100. In each borrowing 2 times of the next place value is borrowed.
When borrowing passes through zeroes then it leaves 1in each place of zero.

Table 4.10: Subtraction of binary number borrowing group of 2_{10}

5th	4th	3rd	2nd	1st	Column Count	
	0	*2*			*When Borrowed 1 from 4th to 3rd column then the remaining bit/bits in the 3rd and 4th column are 2, 0 respectively*	
		0	*1*	*2*	*When Borrowed 1 from 3rd to 1st column then the remaining bit/bits in 1st, 2nd and 3rd column are 2, 1, 0 respectively*	
1	**1**	**1**	**0**	**0**	1st Binary number	Subtraction
– 0	**0**	**1**	**1**	**1**	−2nd Binary number	
1 – 0	0 – 0	2 – 1	1 – 1	2 – 1	*Operation performed*	
1	**0**	**1**	**0**	**1**	Result of subtraction	

Note: When 1 is borrowed from 3rd column to 1st column, then actual value of browed is $2^2 = 4$. In the 2nd column place value of 1 is 2. It gives 0+2+2= 4

From both the tables answer of subtraction is $(10101)_2$

Example-7: $(110011)_2 - (1100)_2$

```
  110011            Decimal equivalent
   −1100                  51
..........              − 12
  100111               ......
                          39
```

4.9 Binary Division:

Division in binary system is same with the decimal system. Only carry and subtraction follow the binary rules.

Exampli-8: Divide 1111 by 101 **Solution:**

```
101 | 1111 | 11
    | 101
    ----------
       101
       101
       ---
       000
```

Answer is $(11)_2$ #

Example-9: $(110011)_2 \div (100)_2$

Solution: <u>**Binary Division**</u> <u>**Decimal equivalent**</u>

```
100 | 110011      | 1100.11
    | 100
-----------------------------
       100
       100
      ---------
          110
          100
      ---------
           100
           100
      --------
             0
```

```
4 | 51 | 12.75
  | 4
-----------------
    11
     8
   -----
     30
     28
   -----
      20
      20
   -----
       0
```

When $(110011)_2$ is divided by $(100)_2$ then the result is $(1100.11)_2$ #

<u>**Note:**</u> $(100)_2 = 4$ and $(110011)_2 = 51$

$$(1100.11)_2 = 1 \times 2^{-2} + 1 \times 2^{-1} + 0 \times 2^{0} + 0 \times 2^{1} + 1 \times 2^{2} + 1 \times 2^{3}$$

$$= 0.25 + 0.5 + 0 + 0 + 4 + 8 = 12.75$$

Example-10: $(110111)_2 \div (111)_2$

Solution:

```
111 | 110111    | 111.10101....
    | 111
------------------------------
      1101
       111
    ---------------
        1100
         111
    ---------------
          1010
           111
    ---------------
            1100
-----------------------------
             111
    ---------------
            1010
             111
    ---------------
   Repeated    11
```

When $(110111)_2$ *is divided by* $(111)_2$ then the result is $(111.10101 \ldots\ldots)_2$

4.10 Technique of writing decimal number in binary form (using place value chart):

The binary bits 1 and 0 have to put in the following chart in such a way that the sum of the product $bit \times place\ value$ become equivalent to the corresponding decimal number. i.e $\sum(bit \times place\ value) =$ **the decimal number.**

Example-11: $(1 \times 16) + (1 \times 8) + (1 \times 4) + (1 \times 2) + (1 \times 1) = 31$

Table 4.11: Decimal number in binary form

Decimal numbers	*Binary place values*							
	64	*32*	*16*	*8*	*4*	*2*	*1*	
1							1	Binary Numbers
2						1	0	
3						1	1	
4					1	0	0	
5					1	0	1	
6					1	1	0	
16			1	0	0	0	0	
17			1	0	0	0	1	
31			1	1	1	1	1	
32		1	0	0	0	0	0	
65	1	0	0	0	0	0	1	

4.11 Uses of Binary Numbers:

All the modern computer technologies are based on this binary system. Computer memory system uses only 0 and 1 in their internal storage. It is used in different complex electronic circuits to manipulate and store all types of data including numbers, words, photos, videos, graphics and music.

Chapter-5

Octal Numbers

5.1 Octal numerals:

The Octal number system is 8-base number system and it uses eight digits from 0 to 7 to represent a number. After 7 the next octal number is 10 and it is equivalent to the decimal number 8 (value is same but the figure is different). In every step of multiple of 8 in decimal, the difference in figure increases by 2 in octal. In the n-th multiple of 8 its increment in octal is 2n. If x is the n-th multiple of 8 in decimal, then the corresponding number in octal is $x + 2n$.

Table 5.1: Octal numbers corresponding to decimals

Decimal	**0**	1	2	3	4	5	6	7	**8**	9	10	11	12	13	14	15
Octal	0	1	2	3	4	5	6	7	**10**	11	12	13	14	15	16	17
Decimal	**16**	17	18	19	20	21	22	23	**24**	25	26	27	28	29	30	31
Octal	20	21	22	23	24	25	26	27	30	31	32	33	34	35	36	37
Decimal	**32**	33	34	35	36	37	38	39	**40**	41	42	43	44	45	46	47
Octal	40	41	42	43	44	45	46	47	50	51	52	53	54	55	56	57
Decimal	**48**	49	50	51	52	53	54	55	**56**	57	58	59	60	61	62	63
Octal	60	61	62	63	64	65	66	67	70	71	72	73	74	75	76	77
Decimal	**64**	65	66	67	68	69	70	71	**72**	73	74	75	76	77	78	79
Octal	80	81	82	83	84	85	86	87	90	91	92	93	94	95	96	97

Table 5.2: Octal numbers corresponding to the multiple of 8 in decimal

Multiples of 8 in decimal	8	16	24	32	40	48	56	64	72	80
Corresponding numbers in octal	10	20	30	40	50	60	70	80	90	100

5.2 Place value of digit in Octal Number System:

Place values of digits in octal number are

$\ldots\ldots, 8^4, 8^3, 8^2, 8^1, 8^0, 8^{-1}, 8^{-2}, 8^{-3}, 8^{-4}, \ldots$

The integral part is from right to left beginning from $8^0 = 1$ and fractional part from left to right beginning from 8^{-1}. Depending on these place values conversion from octal to decimal and from decimal to octal are done. Method is similar to binary system.

5.3 Conversion from octal to decimal:

Example-1: Convert the octal number $(2376)_8$ to decimal form.

Solution: $(2376)_8 = 2 \times 8^3 + 3 \times 8^2 + 7 \times 8^1 + 6 \times 8^0$

$= 2 \times 512 + 3 \times 64 + 7 \times 8 + 6 \times 1$

$= 1024 + 192 + 56 + 6$

$= (1278)_{10}$

The octal number 2376 is equivalent to the decimal 1278 //

Example-2: Convert the octal number $(4312)_{16}$ to decimal form.

Solution: $(4312)_{16} = 4 \times 8^3 + 3 \times 8^2 + 1 \times 8^1 + 2 \times 8^0$

$= 2048 + 192+8+1$

$= (2249)_{10}$

The octal number 4312 is equivalent to the decimal 2249 //

5.4 Conversion from Decimal to Octal:

Successively divide the decimal number by 8 until the value becomes 0 and in each division count the remainder. Next, arrange the remainders beginning ***from bottom to top*** in horizontal line to have the corresponding octal number.

Example-3: Convert the decimal number 307 to octal form

Solution: Let us divide the decimal number 307 successively by 8 and count the remainder.
$307 \div 8 = 38$ Plus remainder 3
$38 \div 8 = 4$ plus remainder 6
$4 \div 8 = 0$ Plus remainder 4
Hence, Octal equivalent of $(307)_{10}$ is $(463)_8$
Therefore, $307 = (463)_8$

Example-4: Convert the decimal number 580 to octal form

Solution: Let us divide the decimal number 307 successively by 8 and count the remainder.
$580 \div 8 = 72$ Plus remainder 4
$72 \div 8 = 9$ plus remainder 0
$9 \div 8 = 1$ Plus remainder 1
$1 \div 8 = 0$ Plus remainder 1
Hence, Octal equivalent of $(580)_{10}$ is $(1104)_8$
That is $580 = (1104)_8$

5.5 Conversion of Decimal fraction to Octal:

First, successive multiplications is applied to a decimal fraction by 8 (octal base) and the counts of integral parts is kept separately in each multiplication. The successive multiplication terminates when decimal part becomes 0 (zero). Next, the integral parts ***beginning from top to bottom*** are arranged in horizontal line to get the octal fraction.

Example-5: Convert $(0.254)_{10}$ to octal decimal.

$0.254 \times 8 = 2.032$, integral part is 2
$0.032 \times 8 = 0.256$, integral part is 0
$0.256 \times 8 = 2.048$, integral part is 2
$0.048 \times 8 = 0.384$, integral part is 0
$0.384 \times 8 = 3.072$, integral part is 3
$0.072 \times 8 = 0.576$, integral part is 0
$0.576 \times 8 = 4.608$, integral part is 4

....

Therefore, $(0.254)_{10} = (0.2020304\ldots)_8$ //

Example-6: Convert $(0.625)_{10}$ to octal decimal.
Solution: $0.625 \times 8 = 5.00$, integral part is 5
Since decimal fraction is 0.00 successive multiplication terminates.
Required result is $(0.625)_{10} = (0.5)_8$

5.6 Conversion from octal to binary:

The base of the octal number is $8 = 2^3$. Therefore, each digit of octal can be replaced by a group of 3 bits in binary numbers. Then their combination gives the respective binary number. The binary equivalent of each octal digit is presented in the following table 5.3.

Table 5.3: Three bits binary Symbols for octal digits

Octal Symbol	0	1	2	3	4	5	6	7
Binary equivalent	000	001	010	011	100	101	110	111

Example-7: Convert $(207)_8$, $(36)_8$ and $(534)_8$ in binary number.

$(207)_8 = (010\ 000\ 111)_2 = (010000111)_2$

$(36)_8 = (011\ 110)_2 = (011110)_2$

$(534)_8 = (101\ 011\ 100)_2 = (101011100)_2$

5.7 Conversion from binary to octal:

First, arrange the 3-bits group from the right to toward left of the binary number. If the leftmost group does not have enough digits to make a 3-bits group, add extra zeros as needed to the left. Convert each binary 3-bit as one octal digit as follows.

Example-8: Convert $(10001101100010111)_2$ to octal number

Solution: Let us arrange the 3-bit group of the binary number as follows

$(10\ 001\ 101\ 100\ 010\ 111)_2$

From right, 111=7, 010=2, 100= 4, 101= 5, 001= 1, 010= 2

Now, $(010\ 001\ 101\ 100\ 010\ 111)_2 = (215427)_8$

Example-9: Convert $(1111101010101)_2$ to octal number

Solution: Let us arrange the 3-bit group of the binary number as follows

$(1\ 111\ 101\ 010\ 101)_2$

From right 101= 5, 010= 2, 101= 5, 111=7, 001=1

Now, $(001\ 111\ 101\ 010\ 101)_2 = (17525)_8$

5.8 Octal Arithmetic:

The octal addition, subtraction, multiplication and division are discussed elaborately in this section.

Table5.4: Octal Addition Table

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

5.8.1 Octal Addition:

Octal addition is performed in the same way of decimal and binary system. Difference is found only in carry numbers when column addition is performed following the octal addition rules.

Example-10: Evaluate the addition $(764)_8 + (657)_8$

Solution: For the better understanding of the first readers addition is performed using the following table 5.5.

Table 5.5: Techniques of Octal Addition

No. of Column	**4^{th}**	**3^{rd}**	**2^{nd}**	**1^{st}**		**Decimal Equivalent**
Carry from 2^{nd} to 3^{rd} column		*1*				
Carry from 1^{st} to 2^{nd} column			*1*			
Octal Numbers to be added		**7**	**6**	**4**		**500**
		6	**5**	**7**		**431**
Columns Addition performed		*16*	14	*13*		
Resulting digit to be retain	**1**	**6**	**4**	**3**		**931**

The result of the addition $(764)_8 + (657)_8 = (1643)_8$ #

Example-11: Find the sum of $(667)_8 + (573)_8$

Solution:

$$\begin{array}{r} 667 \\ +573 \\ \hline (1462)_8 \end{array}$$

5.8.2 Octal Subtraction:

Octal subtraction is same with that of other number systems. The difference is found only in borrowed number depending on base of the system. In decimal system the borrowed numbers are group of 10_{10} (borrowed 10 times from one place to the next lower place). In binary system the borrowed numbers are group of 2_{10} which is equivalent to 10_2. In octal system the borrowed numbers are group of 8_{10} which is equivalent to 10_8

Example-12: To subtract $(354)_8$ from $(502)_8$

Solution: For the better understanding of the first readers subtraction is performed using the following table 5.6.

Table 5.6: Techniques of Octal Subtraction

	3^{rd}	2^{nd}	1^{st}	Column Count	
	4	*7*	*10*	*When 1 is Borrowed from 3^{rd} to 1^{st} column then the remaining numbers in 1^{st}, 2^{nd} and 3^{rd} column are 10, 7, 4*	
	5	**0**	**2**	1^{st} Octal Number	Subtraction
–	**3**	**5**	**4**	-2^{nd} Octal Number	
	4 – 3	7 – 5	12 – 4	*Operation performed*	
	1	**2**	**6**	Result of subtraction	

From the above table the result of subtraction is $= (126)_8$

Note: Why carry 7 is in 2^{nd} column? In octal system preceding to the number 10 is 7.

In case of decimal system it would be 9 (preceding to the number 10 is 9)

Example-13: Subtract $(274)_8$ from $(673)_8$
Solution:

	Decimal Equivalent
653	427
−274	-188
....	....
357	239

The answer of subtraction is = $(357)_8$

Description of the above octal subtraction:

1. First 10 8 is borrowed from 2nd to 1st column, which gives
   $$10 + 3 - 4 = 13 - 4 = 7$$
2. Now in 2nd column remaining figure in the place of 5 is 4. Again, 10_8 is borrowed from 3rd to 2nd column, which gives $10 + 4 - 7 = 14 - 7 = 5$
3. Now in 3rd column remaining figure is 5 in place of 6, which gives $5 - 2 = 3$

5.8.3 Octal Multiplication:

The octal multiplication rules are given in the following table 5.7

Table 5.7: Octal Multiplication

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Worked Examples of Octal Multiplication:

Example-14: $(234)_8 \times (403)_8$
Solution:

```
          2 3 4
        × 4 0 3
  ......................
          7 2 4
+       0 0 0 0
    1 1 6 0 0 0
  ......................
    1 1 6 7 2 4
```

$116724 = (116724)_8$, the required result. #

Example-15: $(521)_8 \times (263)_8$

Solution:

```
          5 2 1
        × 2 6 3
  ........................
    ┌     1 7 6 3
  + ┤   3 7 4 6 0
    └ 1 2 4 2 0 0
  ........................
    1 6 5 6 4 3
```

$= (165643)_8$, the required result. #

Example-16: $(675)_8 \times (602)_8$

Solution:

```
          6 7 5
        × 6 0 2
  ........................
    ┌     2 0 1 2
  + ┤     0 0 0 0
    └ 5 1 5 6 0 0
  ........................
      5 1 7 6 1 2
```

$= (517612)_8$, the required result. #

5.8.4 Octal Division:

Octal division rules are same with that of decimal and binary division systems. But, in case of multiplication and subtraction it follows octal rules. The octal divisionis illustrated in the following examples.

Example-17: $(673)_8 \div (27)_8$

Solution:

```
27 | 673 | 23.20541310...
   | 56  |
   ------------------
      113

      105
   ------------------
       60
       56
   ------------------
       200
       164
   ------------------
        140
        134
   ------------------
         40
         27
   ------------------
         110
         105
   ------------------
          30
          27
   ------------------
           1
```

The division result is

$(23.20541310 \ldots.)_8$ #

Example-18: $(776)_8 \div (63)_8$
Solution:

```
63 | 776 | 12
   | 63  |
   -----------
     146
     146
   -----------
       0
```

Division result is 12 #

5.9 Use of Octal Numbers:

The octal number system is widely used in computer programing and digital numbering system for coding of small addressable unit less than 8-bit character. The computing system in computer uses the words 64-bit, 32-bit, 16-bit and these are further divided into 8-bit words. The octal number is also used for coding purpose in aviation sector. But today the 8-bit octal system has been mostly replaced by 16-bit hexadecimal system.

Chapter-6

Hexadecimal Numbers

6.1 Hexadecimal numbers:

The hexadecimal numeral system is a positional numeral system with base-16. It uses 16 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F to represent a number. The letters A, B, C, D, E, F represent the values 10, 11, 12, 13, 14, 15 respectively. Hexadecimal numbers are widely used by the computer programmers and designers in computer system. Because, a big binary-coded values can be represented by a small hexadecimal number. In binary system an 8-bit byte (8-digit number) can have values ranging from 00000000 to 11111111. Hexadecimal system converted this range from 00 to FF. Each byte is broken into two 4-bit values and represented by two hexadecimal digits.

Place values of the digits in hexadecimal number are

……., 16^4, 16^3, 16^2, 16^1, 16^0, 16^{-1}, 16^{-2}, 16^{-3}, 16^{-4}, …..

Converter Table 6.1:

Binary ↔ Hexadecimal & Decimal ↔ Hexadecimal

Converter of 4 binary bits to 1 hexadecimal digit	
Binary 4-bits	Hex. 1 digit
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	A
1011	B
1100	C
1101	D
1110	E
1111	F

Converter from Decimal to Hexadecimal		Converter from Decimal to Hexadecimal	
Decimal	Hexadecimal	Decimal	Hexadecimal
0	0	……	………
1	1	25	19
2	2	26	1A
3	3	27	1B
4	4	28	1C
5	5	29	1D
6	6	30	1E
7	7	31	1F
8	8	32	20
9	9	33	21
10	A	….	…
11	B	41	29
12	C	42	2A
13	D	43	2B
14	E	44	2C
15	F	45	2D
16	10	46	2E
17	11	47	2F
18	12	48	30
19	13	49	31
…..	….	50	32

6.2 Conversion from Binary to Hexadecimal:

How to convert from binary to hexadecimal are illustrated in the following examples.

Example-1: Convert $(10001101100010)_2$ to hexadecimal.

Solution: First, arrange the 4-bits group from the right to toward left of the binary number. If the leftmost group does not have enough digits to make a 4-bits group, then add extra zeros as needed to the left. Convert each binary 4-bit as one hexadecimal digit to get the hexadecimal number.

$(10001101100010)_2 = (0010\ 0011\ 0110\ 0010)_2$

Now, $0010 = 2$, $0011 = 3$, $0110 = 6$, $0010 = 2$

Therefore, $(10001101100010)_2 = (0010\ 0011\ 0110\ 0010)_2 = (2362)_{16}$ #

Example-2: Convert $(11011011001111)_2$ to hexadecimal number.

Solution: $(11011011001111)_2 = (0011\ 0110\ 1100\ 1111)_2 = (36CF)_{16}$

6.3 Conversion from Hexadecimal to Binary:

How to convert from hexadecimal to binary are illustrated in the following example

Example-3: Convert $(7A9CF)_{16}$ to binary number

Solution: Each digit of hexadecimal number can be replaced by a group of 4-bit in binary number. Then their combination gives the respective binary number. The binary equivalent of each digit of the given hexadecimal number $(7A9CF)_{16}$ are-

7=0111, A=1010, 9=1001, C=1100, F=1111

Therefore, $(7A9CF)_{16} = (0111\ 1010\ 1001\ 1100\ 1111)_2 = (01111010100111001111)_2$

Example-4: Convert $(9DACF)_{16}$ to binary number

Solution: The binary equivalent of each digit of the given hexadecimal number $(9DACF)_{16}$ are-

$9 = 1001$, $D = 1101$, $A = 1010$, $C = 1100$, $F = 1111$

Therefore, $(9DACF)_{16} = (1001\ 1101\ 1010\ 1100\ 1111)_2$

$= (10011101101011001111)_2$ #

6.4 Conversion from hexadecimal to octal:

How to convert a hexadecimal number to octal are illustrated in the following example. First, the hexadecimal number is converted to binary number. Next, the binary number is converted to octal number. **Hexadecimal→Binary→ Octal**

Technique:

1. First, convert the hexadecimal digits to 4-bits binary groups.
2. Second, express the hexadecimal number in binary form.
3. Third, convert the binary number to 3-bits binary group.
4. Forth, replace each 3-bits group by octal symbol

Example-5: Convert $(3BDEF)_{16}$ to binary number and then to Octal number.

Solution: The binary equivalent of each digit of the given hexadecimal number $(3BDEF)_{16}$ are-

3=0011, B=1011, D=1101, E=1110, F=1111

Now, $(3BDEF)_{16} = (0011\ 1011\ 1101\ 1110\ 1111)_2$ [Convert to 4-bits group]

$= (00111011110111101111)_2$ [Binary number]

$= (000\ 111\ 011\ 110\ 111\ 101\ 111)_2$ [Convert to 3-bits group]

$= (0736757)_8$ [Replace each 3-bits group by octal symbol]

Example-6: Convert $(5FDCA8)_{16}$ to Octal number.

Solution: The binary equivalent of each digit of the given hexadecimal number $(5FDCA8)_{16}$ are-

$$5 = 0101,\ F = 1111,\ D = 1101,\ C = 1100,\ A = 1010,\ 8 = 1000$$

Now, $(5FDCA8)_{16} = (0101\ 1111\ 1101\ 1100\ 1010\ 1000)_2$

$= (010111111101110010101000)_2$

$= (010\ 111\ 111\ 101\ 110\ 010\ 101\ 000)_2$

$= (27756250)_8$ #

6.5. Hexadecimal Arithmetic:

Hexadecimal addition, multiplication, subtraction and division are discussed in the following using table where necessary.

6.5.1 Hexadecimal Addition:

Hexadecimal addition is performed using the following table 6.2

Table 6.2: Hexadecimal Addition

+	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
2	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11
3	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12
4	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13
5	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14
6	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15
7	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16
8	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17
9	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18
A	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19
B	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A
C	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B
D	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C
E	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D
F	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E

Example-7: A+3 = D
A+A = 14
B+B =16
C+C =18
D+D =1A
E+E =1C
F+F =1E

Example-8: Find the sum of the following hexadecimal numbers
$(3BF)_{16} + (C9D)_{16}$

Solution:

```
    3BF
  + C9D
..........................
   105C = (105C)16    (Answer)
```

Description of addition:

1. First, F+D = 1C . Put C in the result and carry 1 to next column
2. Second, B+9= 14 and 14+ 1 (carry)=15, Put 5 in the result and carry 1
3. Third, 3+C = F and F+1 (carry)=10 .Put 0 in result of 3rd column and carry 1

Example-9: Evaluate $(8ACF)_{16} + (DB9C)_{16}$

Solution:

```
    (8ACF)16
  + (DB9C)16
..........................................
   (1666B)16    Answer #
```

6.5.2 Hexadecimal multiplication: Hexadecimal multiplication is performed using the following table 6.3.

Table 6.3: Hexadecimal Multiplication

X	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
2	0	2	4	6	8	A	C	E	10	12	14	16	18	1A	1C	1E
3	0	3	6	9	C	F	12	15	18	1B	1E	21	24	27	2A	2D
4	0	4	8	C	10	14	18	1C	20	24	28	2C	30	34	38	3C
5	0	5	A	F	14	19	1E	23	28	2D	32	37	3C	41	46	4B
6	0	6	C	12	18	1E	24	2A	30	36	3C	42	48	4E	54	5A
7	0	7	E	15	1C	23	2A	31	38	3F	46	4D	54	5B	62	69
8	0	8	10	18	20	28	30	38	40	48	50	58	60	68	70	78
9	0	9	12	1B	24	2D	36	3F	48	51	5A	63	6C	75	7E	87
A	0	A	14	1E	28	32	3C	46	50	5A	64	6E	78	82	8C	96
B	0	B	16	21	2C	37	42	4D	58	63	6E	79	84	8F	9A	A5
C	0	C	18	24	30	3C	48	54	60	6C	78	84	90	9C	A8	B4
D	0	D	1A	27	34	41	4E	5B	68	75	82	8F	9C	A9	B6	C3
E	0	E	1C	2A	38	46	54	62	70	7E	8C	9A	A8	B6	C4	D2
F	0	F	1E	2D	3C	4B	5A	69	78	87	96	A5	B4	C3	D2	E1

Note: Addition and Multiplication can be performed using Hexadecimal calculator. However, addition can be performed by figure count.

Example-10: Find the value of $(6CF)_{16} \times (AB)_{16}$

Solution:

```
      6 C F
    × A B
.........................
    4 A E 5
  4 4 1 6 0
...................
  4 8 C 4 5
```

(Result)= $(48C45)_{16}$ #

Description of Multiplication:

1. B × F= A5, Put 5 in the 1st column of result and carry A to the next column.
1. B× C = 84+ A (carry) = 8E, put E in the result and carry 8 to the next column.
2. B× 6 = 42 + 8 (carry) = 4A, put in the result.
3. A× F= 96, Put 6 in the 1st column of result and carry 9 to the next column.
4. A× C = 78+ 9 (carry) = 81, put 1 in the result and carry 8 to the next column
5. A× 6 = 3C+ 8 (carry) = 44, put in the result.
6. 4AE5 + 44160= 48C45

Example-11: Evaluate $(F9DC)_{16} \times (7BE)_{16}$

Solution:

$$(F9DC)_{16}$$
$$\times\ (7BE)_{16}$$

$F9DC \times E =$ DAA08

$F9DC \times B =$ ABC740

$F9DC \times 7 =$ 6D50400

Adding = 78E7548

Answer= $(78E7548)_{16}$ #

6.5.3 Subtraction of Hexadecimal numbers:

Example-12: Find the value of $(30CD)_{16} - (21EF)_{16}$

Solution:

```
 30CD
-21EF
.........................
 0EDE = (EDE)16, Result.     #
```

Description of Subtraction:

1. D is smaller digit than F. Carry 10_{16} from C to D. Now, 10+D−F= 1D−F=E
2. After carry in the place of C, there is C−1=B. Now, B is smaller than E. Next number is 0. So, carry 100_{16} from 3. Now remaining number in 2nd, 3rd, and 4th places are B+10, F, and 2 respectively (Why F? In hexadecimal Preceding to the number 10 is F, but in decimal preceding to the number 10 is 9).
3. Next, B+10−E=1B−E=D
4. In the 3rd column (carry) $F - 1 = E$
5. In the 4th column $2 - 2 = 0$

Example-13: Evaluate $(9FAB)_{16} - (6EF)_{16}$

Solution:

```
 9FAB
-6EF
.........................
  98BC = (98BC)16, Result.   #
```

6.5.4 Division of Hexadecimal numbers:

Divisions of hexadecimal numbers are illustrated in the following examples. The procedures are same with other numeral system. Only addition and multiplication follow the octal rules.

Example-14: Divide $(5DF)_{16}$ by $(1A)_{16}$

Solution: $(5DF)_{16} \div (1A)_{16}$

```
1A ) 5DF ( 39
     4E
.................
      FF
      EA
....................
      15
```

Answer 39 and remainder 15 #

Example-15: Divide
$(9BC)_{16}$ by $(F)_{16}$

Solution: $(9BC)_{16} \div (F)_{16}$

```
F ) 9BC ( A6
    96
 .................
     5C
     5A
 ....................
      2
```

Answer 39 and remainder 2 #

6.6 Use of Hexadecimal Numbers:

Hexadecimal numbers are widely used in computer programming as a convenient notation for the construction of integral values for a large bit binary numbers. In general the output string to binary is very large in their length. Since $2^4 = 16$, therefore conversion to hexadecimal number reduces the binary bit-length by 4 times. The common uses of hexadecimal numbers are-

1. To represent location in computer memory in the internal storage.
2. To represent various colours in the web pages.
3. To represent Media Access Control (MAC) address in a computer networking system.
4. To display error message in the computer.

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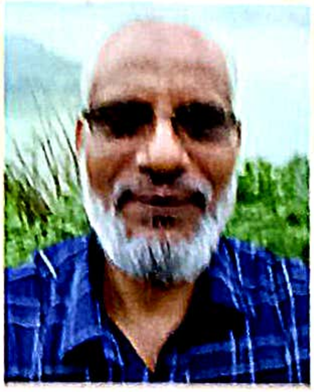

Dr. Abdul Wahed

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